4. EULER EQUATIONS IN THE FOUR-DIMENSIONAL SPACE

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4.1 EULER EQUATIONS IN THE FOUR-DIMENSIONAL SPACE

Known calculations, made as early as by Poisson for the transformation of the velocity of points of space from system Θ to Θ' taking into consideration the four-dimensional space, will be repeated in this chapter.

However, there is a significant difference between the calculations presented herein and those currently existing, since the former strictly introduce differential forms, defined on a differential manifold and connected with the rotations of any SO(n) group, while emphasizing their invariant hypersurfaces.

Let us consider subset U of the differential manifold of physical space M^{Re}, embedded in a four-dimensional Euclidean space E^4. In addition, later in this chapter, we will assume that identity mapping from the subset of space $U \subset E^4$ to U is a map of the subset U.

In system Θ, there is a discriminated point P_0 and the orthonormal base $\{e_i : i = 1,2,3,4\}$ while in system Θ', we have P'_0 and the orthonormal base $\{e'_i : i = 1,2,3,4\}$. The figure below shows both of the coordinate systems and position vectors of particle P. Versors of orientation for mirror spaces α and β are denoted as ${}^{\alpha}\hat{n}$ and ${}^{\beta}\hat{n}$, respectively.

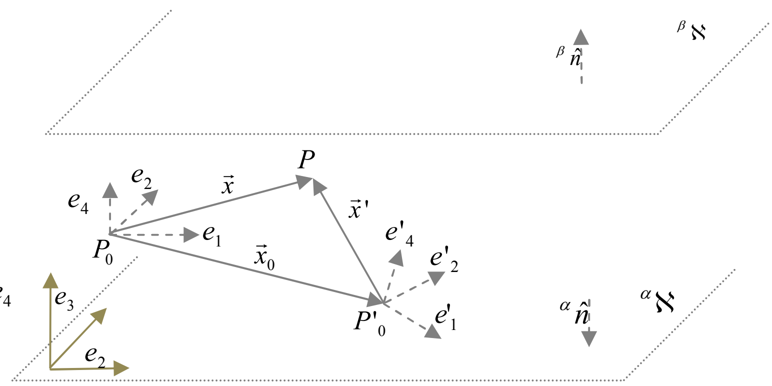

FIG. 1

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\left(\vec{x}_0 + \vec{x}'\right) = v_0 + \frac{d}{dt'}\left(\sum_{i=1}^{4} x'_i e'_i\right)\frac{dt'}{dt} = v_0 + \sum_{i=1}^{4}\frac{dx'_i}{dt'}e'_i + \sum_{i=1}^{4} x'_i \frac{de}{d} \qquad \textbf{(1)}$$

After resolving the quantity $\frac{de'_i}{dt'}$ in the base $\{e'_i : i = 1,2,3,4\}$, we have:

$$\frac{de'_i}{dt'} = \sum_{j=1}^{4} A^i_j e'_j = *^T\left(A^i_1 e'_2 \wedge e'_3 \wedge e'_4 + A^i_2 e'_4 \wedge e'_3 \wedge e'_1 + A^i_3 e'_4 \wedge e'_1 \wedge e'_2 + A^i_4 e'_2 \wedge e'_1 \wedge e'_3\right) \qquad \textbf{(2)}$$

, where A_k^i:

$$A_k^i = \left(e'_k \mid \frac{de'_i}{dt'} \right) \tag{3}$$

, and an isomorphism $*^T$ between the n-dimensional vector space E^n and the $\binom{n}{n-1}$-dimensional space of the external products of $n-1$ vectors.

In a generalized case, an isomorphism $*^T$ between the linear space k of external products $\wedge^k E^n$ and the linear space of n-k external products $\wedge^{n-k} E^n$

$$*^T \left(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \right) = \varepsilon_{i_{k+1}, i_{k+2}, \ldots, i_{n-k}, i_1, i_2, \ldots, i_k} e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}} \tag{4}$$

In particular, for a four-dimensional space, we have:

$$\begin{aligned} *^T \left(e_2 \wedge e_3 \wedge e_4 \right) &= e_1 \\ *^T \left(e_4 \wedge e_3 \wedge e_1 \right) &= e_2 \\ *^T \left(e_4 \wedge e_1 \wedge e_2 \right) &= e_3 \\ *^T \left(e_2 \wedge e_1 \wedge e_3 \right) &= e_4 \end{aligned} \tag{5}$$

Therefore, assuming the following notations:

$$\begin{aligned} e_2 \wedge e_3 \wedge e_4 &= \overset{3}{e_1} \\ e_4 \wedge e_3 \wedge e_1 &= \overset{3}{e_2} \\ e_4 \wedge e_1 \wedge e_2 &= \overset{3}{e_3} \\ e_2 \wedge e_1 \wedge e_3 &= \overset{3}{e_4} \end{aligned} \tag{6}$$

, we have:

$$*^T : \sum_{i=1}^{n} x_i \overset{n-1}{e_i} \rightarrow \vec{x} = \sum_{i=1}^{n} x_i e_i \tag{7}$$

, where $n = 4$, $\vec{x} \in E^n$, $\overset{n-1}{e_i} = \underbrace{e_{i_1} \wedge e_{i_2} \ldots \wedge e_{i_{n-1}}}_{n-1}$

, or more generally,

$$*^T \overset{n-1}{\xi} = \sum_{i=1}^{n} x_i *^T \overset{n-1}{e}_i = \sum_{i=1}^{n} x_i e_i \tag{8}$$

Let us explain that operator $*^T$ is not a Hodge operator because it has a slightly different definition. The Hodge operator is defined as follows:

$$*\left(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \right) = \varepsilon_{i_1, i_2, \ldots, i_k, i_{k+1}, i_{k+2}, \ldots, i_{n-k}} e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}} \tag{9}$$

Thus, the operators have different signs:

$$*\left(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \right) = (-1)^{k(n-k)} *^T \left(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \right) \tag{10}$$

The operator $*^T$ will be referred to as the transposed Hodge star operator and it has the following features:

$$*^T *^T\left(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}\right) = *^T \varepsilon_{i_{k+1},i_{k+2},\ldots,i_{n-k},i_1,i_2,\ldots,i_k} e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}} = \varepsilon_{i_1,i_2,\ldots,i_k,i_{k+1},i_{k+2},\ldots,i_{n-k}} \varepsilon_{i_{k+1},i_{k+2},\ldots,i_{n-k},i_1,i_2,\ldots,i_k} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$$
$$*^T *^T = (-1)^{k(n-k)} Id$$

$$\left(*^T\right)^{-1}\left(e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}}\right) = \varepsilon_{i_{k+1},i_{k+2},\ldots,i_{n-k},i_1,i_2,\ldots,i_k} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} = *(e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}}) =$$
$$= (-1)^{k(n-k)} \varepsilon_{i_1,i_2,\ldots,i_k,i_{k+1},i_{k+2},\ldots,i_{n-k}} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$$
$$\left(*^T\right)^{-1} = *$$

$$*^T *\left(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}\right) = *^T \varepsilon_{i_1,i_2,\ldots,i_k,i_{k+1},i_{k+2},\ldots,i_{n-k}} e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}} = \varepsilon^2{}_{i_1,i_2,\ldots,i_k,i_{k+1},i_{k+2},\ldots,i_{n-k}} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$$
$$*^T * = * *^T = Id \tag{11}$$

$$\left(*^T\right)^{-1} *\left(e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}}\right) = \left(*^T\right)^{-1} \varepsilon_{i_{k+1},i_{k+2},\ldots,i_{n-k},i_1,i_2,\ldots,i_k} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} = \varepsilon_{i_{k+1},i_{k+2},\ldots,i_{n-k},i_1,i_2,\ldots,i_k} \varepsilon_{i_1,i_2,\ldots,i_k,i_{k+1},i_{k+2},\ldots,i_{n-k}} e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}} =$$
$$= (-1)^{k(n-k)} e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}}$$
$$\left(*^T\right)^{-1} * = (-1)^{k(n-k)} Id$$

$$\left(*^T\right)^{-1} *^{-1}\left(e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}}\right) = \left(*^T\right)^{-1} \varepsilon_{i_1,i_2,\ldots,i_k,i_{k+1},i_{k+2},\ldots,i_{n-k}} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} = \varepsilon^2{}_{i_1,i_2,\ldots,i_k,i_{k+1},i_{k+2},\ldots,i_{n-k}} e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_{n-k}}$$
$$\left(*^T\right)^{-1} *^{-1} = *^{-1}\left(*^T\right)^{-1} = Id$$

Using the orthonormality of versors $\{e'_i : i = 1,2,3,4\}$, we have:

$$\left(e'_i \mid e'_j\right) = \delta_{ij} \tag{12}$$

, therefore, differentiating with respect to time in system Θ, we have:

$$\left(\frac{de'_i}{dt'} \mid e'_j\right) + \left(e'_i \mid \frac{de'_j}{dt'}\right) = 0 \tag{13}$$

, from which it follows that $\frac{de'_i}{dt'}$ is orthogonal to e'_i.

Therefore:

$$*\frac{de'_1}{dt'} = A^1_1 e'_2 \wedge e'_3 \wedge e'_4 + A^1_2 e'_4 \wedge e'_3 \wedge e'_1 + A^1_3 e'_4 \wedge e'_1 \wedge e'_2 + A^1_4 e'_2 \wedge e'_1 \wedge e'_3 =$$
$$= \left(e'_1 | \frac{de'_1}{dt'}\right) e'_2 \wedge e'_3 \wedge e'_4 + \left(e'_2 | \frac{de'_1}{dt'}\right) e'_4 \wedge e'_3 \wedge e'_1 + \left(e'_3 | \frac{de'_1}{dt'}\right) e'_4 \wedge e'_1 \wedge e'_2 + \left(e'_4 | \frac{de'_1}{dt'}\right) e'_2 \wedge e'_1 \wedge e'_3 =$$
$$= \left(e'_2 | \frac{de'_1}{dt'}\right) e'_4 \wedge e'_3 \wedge e'_1 - \left(e'_1 | \frac{de'_3}{dt'}\right) e'_4 \wedge e'_1 \wedge e'_2 + \left(e'_4 | \frac{de'_1}{dt'}\right) e'_2 \wedge e'_1 \wedge e'_3 = \tag{14}$$
$$= \left(e'_2 | \frac{de'_1}{dt'}\right) e'_4 \wedge e'_3 \wedge e'_1 + \left(e'_1 | \frac{de'_3}{dt'}\right) e'_4 \wedge e'_2 \wedge e'_1 + \left(e'_4 | \frac{de'_1}{dt'}\right) e'_3 \wedge e'_2 \wedge e'_1 =$$
$$\left(\left(e'_2 | \frac{de'_3}{dt'}\right) e'_1 \wedge e'_4 + \left(e'_3 | \frac{de'_4}{dt'}\right) e'_1 \wedge e'_2 + \left(e'_4 | \frac{de'_2}{dt'}\right) e'_1 \wedge e'_3 + \left(e'_2 | \frac{de'_1}{dt'}\right) e'_4 \wedge e'_3 + \left(e'_1 | \frac{de'_3}{dt'}\right) e'_4 \wedge e'_2 + \left(e'_4 | \frac{de'_1}{dt'}\right) e'_3 \wedge e'_2\right) \wedge e'_1$$

$$*\frac{de'_2}{dt'} = A^2_1 e'_2 \wedge e'_3 \wedge e'_4 + A^2_2 e'_4 \wedge e'_3 \wedge e'_1 + A^2_3 e'_4 \wedge e'_1 \wedge e'_2 + A^2_4 e'_2 \wedge e'_1 \wedge e'_3 =$$
$$= \left(e'_1 | \frac{de'_2}{dt'}\right) e'_2 \wedge e'_3 \wedge e'_4 + \left(e'_2 | \frac{de'_2}{dt'}\right) e'_4 \wedge e'_3 \wedge e'_1 + \left(e'_3 | \frac{de'_2}{dt'}\right) e'_4 \wedge e'_1 \wedge e'_2 + \left(e'_4 | \frac{de'_2}{dt'}\right) e'_2 \wedge e'_1 \wedge e'_3 =$$
$$= \left(e'_1 | \frac{de'_2}{dt'}\right) e'_2 \wedge e'_3 \wedge e'_4 - \left(e'_2 | \frac{de'_3}{dt'}\right) e'_4 \wedge e'_1 \wedge e'_2 + \left(e'_4 | \frac{de'_2}{dt'}\right) e'_2 \wedge e'_1 \wedge e'_3 = \tag{15}$$
$$= \left(e'_1 | \frac{de'_2}{dt'}\right) e'_3 \wedge e'_4 \wedge e'_2 + \left(e'_2 | \frac{de'_3}{dt'}\right) e'_1 \wedge e'_4 \wedge e'_2 + \left(e'_4 | \frac{de'_2}{dt'}\right) e'_1 \wedge e'_3 \wedge e'_2 =$$
$$\left(\left(e'_1 | \frac{de'_2}{dt'}\right) e'_3 \wedge e'_4 + \left(e'_2 | \frac{de'_3}{dt'}\right) e'_1 \wedge e'_4 + \left(e'_4 | \frac{de'_2}{dt'}\right) e'_1 \wedge e'_3\right) \wedge e'_2$$

$$*\frac{de'_3}{dt'}=A^3_1e'_2\wedge e'_3\wedge e'_4+A^3_2e'_4\wedge e'_3\wedge e'_1+A^3_3e'_4\wedge e'_1\wedge e'_2+A^3_4e'_2\wedge e'_1\wedge e'_3=$$
$$=\left(e'_1|\frac{de'_3}{dt'}\right)e'_2\wedge e'_3\wedge e'_4+\left(e'_2|\frac{de'_3}{dt'}\right)e'_4\wedge e'_3\wedge e'_1+\left(e'_3|\frac{de'_3}{dt'}\right)e'_4\wedge e'_1\wedge e'_2+\left(e'_4|\frac{de'_3}{dt'}\right)e'_2\wedge e'_1\wedge$$
$$=\left(e'_1|\frac{de'_3}{dt'}\right)e'_4\wedge e'_2\wedge e'_3+\left(e'_2|\frac{de'_3}{dt'}\right)e'_1\wedge e'_4\wedge e'_3+\left(e'_3|\frac{de'_4}{dt'}\right)e'_1\wedge e'_2\wedge e'_3= \tag{16}$$
$$\left(\left(e'_1|\frac{de'_3}{dt'}\right)e'_4\wedge e'_2+\left(e'_2|\frac{de'_3}{dt'}\right)e'_1\wedge e'_4+\left(e'_3|\frac{de'_4}{dt'}\right)e'_1\wedge e'_2\right)\wedge e'_3$$

$$*\frac{de'_4}{dt'}=A^4_1e'_2\wedge e'_3\wedge e'_4+A^4_2e'_4\wedge e'_3\wedge e'_1+A^4_3e'_4\wedge e'_1\wedge e'_2+A^4_4e'_2\wedge e'_1\wedge e'_3=$$
$$=\left(e'_1|\frac{de'_4}{dt'}\right)e'_2\wedge e'_3\wedge e'_4+\left(e'_2|\frac{de'_4}{dt'}\right)e'_4\wedge e'_3\wedge e'_1+\left(e'_3|\frac{de'_4}{dt'}\right)e'_4\wedge e'_1\wedge e'_2+\left(e'_4|\frac{de'_4}{dt'}\right)e'_2\wedge e'_1\wedge$$
$$=\left(e'_4|\frac{de'_1}{dt'}\right)e'_3\wedge e'_2\wedge e'_4+\left(e'_4|\frac{de'_2}{dt'}\right)e'_1\wedge e'_3\wedge e'_4+\left(e'_3|\frac{de'_4}{dt'}\right)e'_1\wedge e'_2\wedge e'_4= \tag{17}$$
$$=\left(\left(e'_4|\frac{de'_1}{dt'}\right)e'_3\wedge e'_2+\left(e'_4|\frac{de'_2}{dt'}\right)e'_1\wedge e'_3+\left(e'_3|\frac{de'_4}{dt'}\right)e'_1\wedge e'_2\right)\wedge e'_4$$

The repeated bivector will be referred to as angular velocity $\overset{2}{\omega}$ and will be defined as follows:

$$\overset{2}{\omega}:=\left(e'_2|\frac{de'_3}{dt'}\right)e'_1\wedge e'_4+\left(e'_3|\frac{de'_4}{dt'}\right)e'_1\wedge e'_2+\left(e'_4|\frac{de'_2}{dt'}\right)e'_1\wedge e'_3+$$
$$+\left(e'_1|\frac{de'_2}{dt'}\right)e'_3\wedge e'_4+\left(e'_3|\frac{de'_1}{dt'}\right)e'_2\wedge e'_4+\left(e'_1|\frac{de'_4}{dt'}\right)e'_2\wedge e'_3 \tag{18}$$

It is no accident that angular velocity $\overset{2}{\omega}$ was denoted by us as a differential form of second order, defined on the tangent bundle TM^{Re} of manifold M^{Re} of the four-dimensional physical space because, according to the definition, a differential form at point q of manifold M^{Re} is an external k-form, defined on the tangent space TM_q^{Re}.[1]

Therefore, the equation of angular velocity $\overset{2}{\omega}$ may be written in a generalized way as a differential form of second order, defined on the manifold TM^4 in the following form:

[1] See the supplement. We wish to explain that the passage from vectors to linear forms is possibile due to the isomorphism, defined by the scalar product. The scalar product may be defined by means of Riemann metric on Reimann manifold.

$$\overset{2}{\omega}(q) := \omega_{1,4}(q)dq_1 \wedge dq_4 + \omega_{1,2}(q)dq_1 \wedge dq_2 + \omega_{1,3}(q)dq_1 \wedge dq_3 +$$
$$+ \omega_{3,4}(q)dq_3 \wedge dq_4 + \omega_{2,4}(q)dq_2 \wedge dq_4 + \omega_{2,3}(q)dq_2 \wedge dq_3$$
$$\omega_{1,4}(q) = \left(e'_2(q) \,|\, \frac{de'_3}{dt'}(q) \right) \overset{def}{=} \omega_1(q)$$
$$\omega_{1,2}(q) = \left(e'_3(q) \,|\, \frac{de'_4}{dt'}(q) \right) \overset{def}{=} \omega_4(q)$$
$$\omega_{1,3}(q) = \left(e'_4(q) \,|\, \frac{de'_2}{dt'}(q) \right) \overset{def}{=} \omega_5(q) \tag{19}$$
$$\omega_{3,4}(q) = \left(e'_1(q) \,|\, \frac{de'_2}{dt'}(q) \right) \overset{def}{=} \omega_3(q)$$
$$\omega_{2,4}(q) = \left(e'_3(q) \,|\, \frac{de'_1}{dt'}(q) \right) \overset{def}{=} \omega_2(q)$$
$$\omega_{2,3}(q) = \left(e'_1(q) \,|\, \frac{de'_4}{dt'}(q) \right) \overset{def}{=} \omega_6(q)$$

, where $q \in M^4$

This means that we are going to use the external and differential form of angular velocity, which is expected to prevent any misunderstanding. Moreover, the external form of the equation is defined on a $\binom{n}{n-2}$-dimensional linear space of $\wedge^{n-2} E^n$ external forms of the order n-2 defined on space E^n.

Reassuming, in the physical space which is a n-dimensional differential manifold $n > 3$ angular velocity is a differential form of the order n-2, defined on the manifold.

Ultimately, by substituting in Formula (1), we have:

$$\frac{d\vec{x}}{dt} = \vec{v}_0 + \frac{d\vec{x}'}{dt} + *^T \overset{2}{\omega} \wedge \vec{x}' \tag{20}$$

, and:

$$\vec{v} = \vec{v}_0 + \vec{v}' + *^T \overset{2}{\omega} \wedge \vec{x}' \tag{21}$$

In the coordinate system Θ' we will examine, in every case, only such elements[2], for which velocity $\vec{v}'$ is equal to zero:

$$\vec{v} = \vec{v}_0 + *^T \overset{2}{\omega} \wedge \vec{x}' \tag{22}$$

More generally, in a n-dimensional space, we have:

$$\vec{v} = \vec{v}_0 + *^T \overset{n-2}{\omega} \wedge \vec{x}' \tag{23}$$

, but $\vec{v}_0$ may be written as:

$$\vec{v}_0 = *^T * \vec{v}_0 \tag{24}$$

[2] i.e., small regions of space.

Therefore, after substituting (24) in Formula (23), we have:

$$\vec{v} = \vec{v}_0 + *^T \overset{n-2}{\omega} \wedge \vec{x}' = *^T * \vec{v}_0 + *^T \overset{n-2}{\omega} \wedge \vec{x}' = \\ = *^T \left(* \vec{v}_0 + \overset{n-2}{\omega} \wedge \vec{x}' \right) \tag{25}$$

We will frequently describe such cases in which $\vec{v}_0 = 0$. For such cases, Formula (23) is in a simplified form:

$$\vec{v} = *^T \overset{n-2}{\omega} \wedge \vec{x}' \tag{26}$$

Let us note that for every element $\vec{x}'$ of n -dimensional space $\vec{x}' \in E^n$ we have:

$$\overset{n-2}{\omega} \to \frac{d\vec{x}}{dt} \equiv *^T \overset{n-2}{\omega} \wedge \vec{x}' \tag{27}$$

Let us write down the results of the effect of isomorphism $*^T$ on the base versors of bivectors from a six dimensional space $\overset{2}{e_i} \in \wedge^2 E^4$:

$$\begin{aligned}
*^T \left(\overset{2}{e}_1 \wedge \sum_{i=1}^{4} x_i e_i \right) &= *^T \left(e_1 \wedge e_4 \wedge \sum_{i=1}^{4} x_i e_i \right) = -x_2 e_3 + x_3 e_2 \\
*^T \left(\overset{2}{e}_2 \wedge \sum_{i=1}^{4} x_i e_i \right) &= *^T \left(e_2 \wedge e_4 \wedge \sum_{i=1}^{4} x_i e_i \right) = x_1 e_3 - x_3 e_1 \\
*^T \left(\overset{2}{e}_3 \wedge \sum_{i=1}^{4} x_i e_i \right) &= *^T \left(e_3 \wedge e_4 \wedge \sum_{i=1}^{4} x_i e_i \right) = -x_1 e_2 + x_2 e_1 \\
*^T \left(\overset{2}{e}_4 \wedge \sum_{i=1}^{4} x_i e_i \right) &= *^T \left(e_1 \wedge e_2 \wedge \sum_{i=1}^{4} x_i e_i \right) = -x_3 e_4 + x_4 e_3 \\
*^T \left(\overset{2}{e}_5 \wedge \sum_{i=1}^{4} x_i e_i \right) &= *^T \left(e_1 \wedge e_3 \wedge \sum_{i=1}^{4} x_i e_i \right) = x_2 e_4 - x_4 e_2 \\
*^T \left(\overset{2}{e}_6 \wedge \sum_{i=1}^{4} x_i e_i \right) &= *^T \left(e_2 \wedge e_3 \wedge \sum_{i=1}^{4} x_i e_i \right) = -x_1 e_4 + x_4 e_1
\end{aligned} \tag{28}$$

The base of the space of bivectors $\wedge^2 E^4$ is the set: $\left\{ \overset{2}{e}_1, \overset{2}{e}_2, \overset{2}{e}_3, \overset{2}{e}_4, \overset{2}{e}_5, \overset{2}{e}_6 \right\}$

, where the following notations have been introduced:

$$\begin{aligned}
\overset{2}{e}_1 &\equiv e_1 \wedge e_4 \\
\overset{2}{e}_2 &\equiv e_2 \wedge e_4 \\
\overset{2}{e}_3 &\equiv e_3 \wedge e_4 \\
\overset{2}{e}_4 &\equiv e_1 \wedge e_2 \\
\overset{2}{e}_5 &\equiv e_1 \wedge e_3 \\
\overset{2}{e}_6 &\equiv e_2 \wedge e_3
\end{aligned} \tag{29}$$

Further, let us introduce Euclidean structure in space $\wedge^{n-2} E^4$, assuming the following definition of a scalar product:

$$\forall_{\vec{x} \in \wedge^{n-2} E^n \wedge \vec{y} \in \wedge^{n-2} E^n} (\vec{x} \mid \vec{y}) = \sum_{i=1}^{\binom{n}{2}} x_i y_i \quad \textbf{(30)}$$

, where:

$$\vec{x} = \sum_{i=1}^{\binom{n}{2}} x_i \overset{n-2}{e}_i$$
$$\vec{y} = \sum_{i=1}^{\binom{n}{2}} y_i \overset{n-2}{e}_i \quad \textbf{(31)}$$

The norm, determined by the scalar product, is in the following form:

$$\|\vec{x}\| = (\vec{x} \mid \vec{x}) = \sum_{i=1}^{\binom{n}{2}} x_i^{\,2} \quad \textbf{(32)}$$

4.2 ANGULAR MOMENTUM IN THE FOUR-DIMENSIONAL SPACE

The following defininition of angular momentum $\overset{n-2}{J} \in \wedge^{n-2} E^n$ is assumed:

$$\overset{n-2}{J} \equiv *\sum_{i=1}^{N} \vec{x}_i \wedge m_i \vec{v}_i = *\sum_{i=1}^{N} m_i \vec{x}_i \wedge \vec{v}_i \tag{33}$$

Using equations obtained earlier, we have:

$$\begin{aligned} \overset{n-2}{J} &= *\sum_{i=1}^{N} m_i \vec{x}_i \wedge \vec{v}_i = *\sum_{i=1}^{N} m_i \left(\vec{x}_0 + \vec{x}'_i\right) \wedge \left(\vec{v}_0 + *^T \overset{2}{\omega} \wedge \vec{x}_i'\right) = \\ &= *\sum_{i=1}^{N} m_i \vec{x}'_i \wedge \vec{v}_0 + *\sum_{i=1}^{N} m_i \vec{x}_0 \wedge *^T \overset{2}{\omega} \wedge \vec{x}_i' + *\sum_{i=1}^{N} m_i \vec{x}'_i \wedge *^T \overset{2}{\omega} \wedge \vec{x}_i' = \\ &= *M\vec{R} \wedge \vec{v}_0 + *M\vec{x}_0 \wedge *^T \overset{2}{\omega} \wedge \vec{R}' + *\sum_{i=1}^{N} m_i \vec{x}'_i \wedge *^T \overset{2}{\omega} \wedge \vec{x}_i' \end{aligned} \tag{34}$$

, where it is assumed that:

$$\begin{aligned} M &\equiv \sum_{i=1}^{N} m_i \\ \vec{R} &\equiv \frac{1}{M} \sum_{i=1}^{N} m_i \vec{x}_i \\ \vec{R}' &\equiv \frac{1}{M} \sum_{i=1}^{N} m_i \vec{x}'_i \end{aligned} \tag{35}$$

Assuming that vectors $\vec{R}$ and $\vec{v}_0$ are coaxial, $\vec{R}' = 0$ and $n = 4$, we have the following angular momentum formula:

$$\overset{2}{J} = *\sum_{i=1}^{N} m_i \vec{x}'_i \wedge *^T \overset{2}{\omega} \wedge \vec{x}_i' \tag{36}$$

Let us use Formulas (28) for calculating the components of angular momentum $\overset{2}{J}$ for $\overset{2}{\omega} = \omega_1 \overset{2}{e_1}' = \omega_1 e_1' \wedge e_4'$ and $\vec{x}_i' = x_{2,i}' e_2' + x_{3,i}' e_3'$[3]:

$$\begin{aligned} \overset{2}{J}_{1,4} &= *\sum_{i=1}^{N} m_i \left(x_{2,i}' e_2' + x_{3,i}' e_3'\right) \wedge *^T \omega_1 \overset{2}{e_1} \wedge \left(x_{2,i}' e_2' + x_{3,i}' e_3'\right) = \\ &= *\sum_{i=1}^{N} m_i \omega_1 \left(x_{2,i}' e_2' + x_{3,i}' e_3'\right) \wedge \left(-x_{2,i}' e_3' + x_{3,i}' e_2'\right) = \\ &= *\sum_{i=1}^{N} m_i \omega_1 \left(\left(x_{2,i}'\right)^2 + \left(x_{3,i}'\right)^2\right) e_2' \wedge e_3' = \sum_{i=1}^{N} m_i \omega_1 \left(\left(x_{2,i}'\right)^2 + \left(x_{3,i}'\right)^2\right) e_1' \wedge e_4' \end{aligned} \tag{37}$$

Let us note that the left side of the equation is a bivector and on the right side there is another bivector, namely angular velocity.

[3] Distance, in directions which are perpendicular to the axis of rotation.

Therefore, we can write:

$$\overset{2}{J} := I \overset{2}{\omega} \tag{38}$$

, where I is a symmetrical linear operator (or tensor of inertia) transforming bivector $\overset{2}{\omega}$ into bivector $\overset{2}{J}$.

More generally, in a n-dimensional space, we have:

$$\overset{n-2}{J} := I \overset{n-2}{\omega} \tag{39}$$

In the matrix form, operator I in the base $\left\{\overset{2}{e_1},\overset{2}{e_2},\overset{2}{e_3},\overset{2}{e_4},\overset{2}{e_5},\overset{2}{e_6}\right\}$ can be written as $I_{i,j}$ where $i \in \left\{1,2,\ldots,\binom{n}{n-2}\right\} \equiv N_{n-2} \wedge j \in N_{n-2} = N_2$.

Given the notations introduced herein, the component of angular momentum $\overset{2}{J_i}$ is expressed by means of the general formula:

$$\begin{aligned} \overset{2}{J} &= \sum_{i=1}^{6} J_i \overset{2}{e_i} = \sum_{i=1}^{6}\sum_{j=1}^{6} I_{i,j}\,\omega_j \overset{2}{e_j} = \sum_{j=1}^{6}\sum_{i=1}^{6} I_{j,i}\,\omega_i \overset{2}{e_i} = \sum_{i=1}^{6}\sum_{j=1}^{6} I_{j,i}\,\omega_i \overset{2}{e_i} \\ J_i &= \sum_{j=1}^{6} I_{j,i}\,\omega_i \end{aligned} \tag{40}$$

, or, more generally:

$$\begin{aligned} \overset{n-2}{J} &= \sum_{i=1}^{N_{n-2}} J_i \overset{n-2}{e_i} = \sum_{i=1}^{N_{n-2}}\sum_{j=1}^{N_{n-2}} I_{i,j}\,\omega_j \overset{n-2}{e_j} = \sum_{j=1}^{N_{n-2}}\sum_{i=1}^{N_{n-2}} I_{j,i}\,\omega_i \overset{n-2}{e_i} = \sum_{i=1}^{N_{n-2}}\sum_{j=1}^{N_{n-2}} I_{j,i}\,\omega_i \overset{n-2}{e_i} \\ J_i &= \sum_{j=1}^{N_{n-2}} I_{j,i}\,\omega_i \end{aligned} \tag{41}$$

Same as in the three-dimensional case, the bivectors of the base in which the operator I is diagonal will be referred to as principal planes of moments of inertia. Moments of inertia with respect to such planes will be referred to as principal moments of inertia and denoted as $I_{N_{n-2}}$.

Using the notation we have introduced for the base bivectors, we have the following for angular velocity:

$$\overset{2}{\omega}=\left(e_2 \,|\, \frac{de_3}{dt}\right)\overset{2}{e_1}+\left(e_3 \,|\, \frac{de_1}{dt}\right)\overset{2}{e_2}+\left(e_1 \,|\, \frac{de_2}{dt}\right)\overset{2}{e_3}+\left(e_3 \,|\, \frac{de_4}{dt}\right)\overset{2}{e_4}+\left(e_4 \,|\, \frac{de_2}{dt}\right)\overset{2}{e_5}+\left(e_1 \,|\, \frac{de_4}{dt}\right)\overset{2}{e_6} \tag{42}$$

Comparing angular velocity in the four-dimensional space and angular velocity in the three-dimensional space:

$$\vec{\omega}=\left(e_3 \,|\, \frac{de_2}{dt}\right)e_1+\left(e_1 \,|\, \frac{de_3}{dt}\right)e_2+\left(e_2 \,|\, \frac{de_1}{dt}\right)e_3 \tag{43}$$

, we find that the respective components of the bivector $\overset{2}{\omega}$ of angular velocity may be equated with the components of the velocity vector. Directly from Formulas (42) and (43) we obtain:

$$\overset{2}{\omega}=-\omega_1\overset{2}{e}_1-\omega_2\overset{2}{e}_2-\omega_3\overset{2}{e}_3+\left(e_3\,|\,\frac{de_4}{dt'}\right)\overset{2}{e}_4+\left(e_4\,|\,\frac{de_2}{dt'}\right)\overset{2}{e}_5+\left(e_1\,|\,\frac{de_4}{dt'}\right)\overset{2}{e}_6 \qquad \textbf{(44)}$$

, and:

$$\overset{2}{\omega}=-\omega_1\overset{2}{e}_1-\omega_2\overset{2}{e}_2-\omega_3\overset{2}{e}_3+\omega_4\overset{2}{e}_4+\omega_5\overset{2}{e}_5+\omega_6\overset{2}{e}_6 \qquad \textbf{(45)}$$

, where:

$$\begin{aligned}\omega_4&=\left(e_3\,|\,\frac{de_4}{dt'}\right)\\ \omega_5&=\left(e_4\,|\,\frac{de_2}{dt'}\right)\\ \omega_6&=\left(e_1\,|\,\frac{de_4}{dt'}\right)\end{aligned} \qquad \textbf{(46)}$$

Therefore, from Formula (45) it follows that the components of the bivector of velocity are rotations in planes which are spread on the complements of the base bivectors. As an example, let us consider component $\overset{2}{e}_1$ of the bivector of velocity $\overset{2}{\omega}$ which is $-\omega_1$ according to Formula (45). This means that the issue dealt with is rotations in a plane which is spread on vectors $L(e_2,e_3)$. The subspace which is spread on vectors $L(e_1,e_4)$, i.e., the same vectors which form the base bivector $\overset{2}{e}_1$ of subspace $\wedge^{n-2}E^n$ where $n=4$, is the invariant subspace of such rotations.

On the other hand, when comparing moments of inertia I_i with their three-dimensional counterparts, we have:

$$\begin{aligned}I_1&=I_{1,1}=I_{14,14}=I_1^{(3d)}\\ I_2&=I_{24,24}=I_2^{(3d)}\\ I_3&=I_{34,34}=I_3^{(3d)}\end{aligned} \qquad \textbf{(47)}$$

Therefore, angular momentum in the principal planes $\left\{\overset{2}{e}{}'_{N_{n-2}}\right\}$ of the moment of inertia takes the form:

$$\begin{aligned}&\overset{2}{J}=\overset{2}{I}\overset{2}{\omega}=\\ &=-I_1\omega_1\overset{2}{e}{}'_1-I_2\omega_2\overset{2}{e}{}'_2-I_3\omega_3\overset{2}{e}{}'_3+I_4\,\omega_4\overset{2}{e}{}'_4+I_5\,\omega_5\overset{2}{e}{}'_5+I_6\,\omega_6\overset{2}{e}{}'_6\end{aligned} \qquad \textbf{(48)}$$

If the angular momentum is a bivector then its time derivative, equated with torque, is also a bivector:

$$\frac{d\overset{2}{J}}{dt} = \overset{2}{D} \qquad (49)$$

More generally, for a n-dimensional space:

$$\frac{d\overset{n-2}{J}}{dt} = \overset{n-2}{D} \qquad (50)$$

For calculating the time derivative of the bivector at the passage to the system Θ' we will use Formula (20) which takes the following form after substituting $P_0 = P'_0$:

$$\left(\frac{d\vec{x}}{dt}\right)_{\{e_1,e_2,e_3,e_4\}} = \left(\frac{d\vec{x}'}{dt} + *^T \overset{2}{\omega} \wedge \vec{x}\right)_{\{e'_1,e'_2,e'_3,e'_4\}} \qquad (51)$$

Transformation of the time derivative of bivector $\overset{2}{u}$ to the system Θ' is in the following form:

$$\frac{d\overset{2}{u}}{dt} = \frac{d\left(\sum_{i=1}^{6} u'_i \overset{2}{e'_i}\right)}{dt} = \sum_{i=1}^{6} \frac{d u'_i}{dt} \overset{2}{e'_i} + \sum_{i=1}^{6} u'_i \frac{d\overset{2}{e'_i}}{dt} \qquad (52)$$

$$\begin{aligned} \frac{d\overset{2}{e'_1}}{dt} &= \frac{d(e'_1 \wedge e'_4)}{dt} = \frac{de'_1}{dt} \wedge e'_4 + e'_1 \wedge \frac{de'_4}{dt} = \\ &= *^T\left(\overset{2}{\omega} \wedge e'_1\right) \wedge e'_4 + e'_1 \wedge *^T\left(\overset{2}{\omega} \wedge e'_4\right) = \overset{2}{\omega} \wedge^* \overset{2}{e'_1} \end{aligned} \qquad (53)$$

, where we have introduced a new operator $\wedge^*$, defined as follows:

$$\overset{2}{u} \wedge^* (x \wedge y) \equiv *^T\left(\overset{2}{u} \wedge x\right) \wedge y + x \wedge *^T\left(\overset{2}{u} \wedge y\right) \qquad (54)$$

, and more generally:

$$\overset{n-2}{u} \wedge^* (x \wedge y) \equiv *^T\left(\overset{n-2}{u} \wedge x\right) \wedge y + x \wedge *^T\left(\overset{n-2}{u} \wedge y\right) \qquad (55)$$

Therefore, after substituting **(52)** in **(51)** we obtain a formula for transformating the time derivative of the bivector:

$$\left(\frac{d\overset{2}{u}}{dt}\right)_{\Theta} = \left(\sum_{i=1}^{6} \frac{d u'_i}{dt} \overset{2}{e'_i} + \overset{2}{\omega} \wedge^* \overset{2}{u}\right)_{\Theta'} \qquad (56)$$

, and more generally:

$$\left(\frac{d\overset{n-2}{u}}{dt}\right)_{\Theta} = \left(\sum_{i=1}^{\binom{n}{n-2}} \frac{d\,u'_i}{dt}\overset{n-2}{e'_i} + \overset{n-2}{\omega}\wedge^{*}\overset{n-2}{u}\right)_{\Theta'} \tag{57}$$

After passing to the system Θ', in which the operator $\overset{2}{I}$ is diagonal and its components are constant in time, we obtain from Formula (49):

$$\begin{aligned}\frac{d\overset{2}{J}}{dt} = \frac{d\left(\overset{2}{I}\overset{2}{\omega}\right)}{dt} &= -I_1\dot{\omega}_1\overset{2}{e'_1} - I_1\omega_1\overset{2}{\omega}\wedge^{*}\overset{2}{e'_1} - I_2\dot{\omega}_2\overset{2}{e'_2} + \\ &- I_2\omega_2\overset{2}{\omega}\wedge^{*}\overset{2}{e'_2} - I_3\dot{\omega}_3\overset{2}{e'_3} - I_3\omega_3\overset{2}{\omega}\wedge^{*}\overset{2}{e'_3} + \\ &+ I_4\dot{\omega}_4\overset{2}{e'_4} + I_4\omega_4\overset{2}{\omega}\wedge^{*}\overset{2}{e'_4} + I_5\dot{\omega}_5\overset{2}{e'_5} + \\ &+ I_5\omega_5\overset{2}{\omega}\wedge^{*}\overset{2}{e'_5} + I_6\dot{\omega}_6\overset{2}{e'_6} + I_6\omega_6\overset{2}{\omega}\wedge^{*}\overset{2}{e'_6}\end{aligned} \tag{58}$$

, or more generally:

$$\frac{d\overset{n-2}{J}}{dt} = \sum_{i=1}^{\binom{n}{2}} - I_i\dot{\omega}_i\overset{n-2}{e'_i} - I_i\omega_i\overset{n-2}{\omega}\wedge^{*}\overset{n-2}{e'_i} \tag{59}$$

After substituting the bivector $\overset{2}{\omega}$ in Equation (56) with the expression from Equation (18) we obtain:

$$\begin{aligned}
&\overset{2}{\omega}\wedge^{*}\overset{2}{e'_1} = *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_1\right)\wedge e'_4 + \\
&e'_1\wedge *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_4\right) = \\
&= -\omega_2 e'_3\wedge e'_4 + \omega_3 e'_2\wedge e'_4 + \omega_4 e'_1\wedge e'_3 - \omega_5 e'_1\wedge e'_2 = -\omega_2\overset{2}{e'_3} + \omega_3\overset{2}{e'_2} + \omega_4\overset{2}{e'_5} - \omega_5\overset{2}{e'_4} \\
&\overset{2}{\omega}\wedge^{*}\overset{2}{e'_2} = *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_2\right)\wedge e'_4 + \\
&e'_2\wedge *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_4\right) = \\
&= \omega_1 e'_3\wedge e'_4 - \omega_3 e'_1\wedge e'_4 + \omega_4 e'_2\wedge e'_3 + \omega_6 e'_2\wedge e'_1 = \omega_1\overset{2}{e'_3} - \omega_3\overset{2}{e'_1} + \omega_4\overset{2}{e'_6} - \omega_6\overset{2}{e'_4} \\
&\overset{2}{\omega}\wedge^{*}\overset{2}{e'_3} = *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_3\right)\wedge e'_4 + \\
&e'_3\wedge *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_4\right) = \\
&= -\omega_1 e'_2\wedge e'_4 + \omega_2 e'_1\wedge e'_4 - \omega_5 e'_3\wedge e'_2 + \omega_6 e'_3\wedge e'_1 = -\omega_1\overset{2}{e'_2} + \omega_2\overset{2}{e'_1} + \omega_5\overset{2}{e'_6} - \omega_6\overset{2}{e'_5} \\
&\overset{2}{\omega}\wedge^{*}\overset{2}{e'_4} = *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_1\right)\wedge e'_2 + \\
&e'_1\wedge *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_2\right) = \\
&= -\omega_2 e'_3\wedge e'_2 - \omega_6 e'_4\wedge e'_2 + \omega_1 e'_1\wedge e'_3 + \omega_5 e'_1\wedge e'_4 = \omega_2\overset{2}{e'_6} + \omega_6\overset{2}{e'_2} + \omega_1\overset{2}{e'_5} + \omega_5\overset{2}{e'_1} \\
&\overset{2}{\omega}\wedge^{*}\overset{2}{e'_5} = *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_1\right)\wedge e'_3 + \\
&e'_1\wedge *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_3\right) = \\
&= \omega_3 e'_2\wedge e'_3 - \omega_6 e'_4\wedge e'_3 - \omega_1 e'_1\wedge e'_2 - \omega_4 e'_1\wedge e'_4 = \omega_3\overset{2}{e'_6} + \omega_6\overset{2}{e'_3} - \omega_1\overset{2}{e'_4} - \omega_4\overset{2}{e'_1} \\
&\overset{2}{\omega}\wedge^{*}\overset{2}{e'_6} = *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_2\right)\wedge e'_3 + \\
&e'_2\wedge *^{T}\left(\left(-\omega_1 e'_1\wedge e'_4 - \omega_2 e'_2\wedge e'_4 - \omega_3 e'_3\wedge e'_4 + \omega_4 e'_1\wedge e'_2 + \omega_5 e'_1\wedge e'_3 + \omega_6 e'_2\wedge e'_3\right)\wedge e'_3\right) = \\
&= -\omega_3 e'_1\wedge e'_3 + \omega_5 e'_4\wedge e'_3 + \omega_2 e'_2\wedge e'_1 - \omega_4 e'_2\wedge e'_4 = -\omega_3\overset{2}{e'_5} - \omega_5\overset{2}{e'_3} - \omega_2\overset{2}{e'_4} - \omega_4\overset{2}{e'_2}
\end{aligned} \tag{60}$$

Therefore, ultimately:

$$
\begin{aligned}
\frac{d\overset{2}{J}}{dt} &= \frac{d\left(\overset{2}{I}\overset{2}{\omega}\right)}{dt} = -I_1\dot{\omega}_1\overset{2}{e}'_1 - I_1\omega_1\left(-\omega_2\overset{2}{e}'_3 + \omega_3\overset{2}{e}'_2 + \omega_4\overset{2}{e}'_5 - \omega_5\overset{2}{e}'_4\right) + \\
&- I_2\dot{\omega}_2\overset{2}{e}'_2 - I_2\omega_2\left(\omega_1\overset{2}{e}'_3 - \omega_3\overset{2}{e}'_1 + \omega_4\overset{2}{e}'_6 - \omega_6\overset{2}{e}'_4\right) + \\
&- I_3\dot{\omega}_3\overset{2}{e}'_3 - I_3\omega_3\left(-\omega_1\overset{2}{e}'_2 + \omega_2\overset{2}{e}'_1 + \omega_5\overset{2}{e}'_6 - \omega_6\overset{2}{e}'_5\right) + \\
&+ I_4\dot{\omega}_4\overset{2}{e}'_4 + I_4\omega_4\left(\omega_2\overset{2}{e}'_6 + \omega_6\overset{2}{e}'_2 + \omega_1\overset{2}{e}'_5 + \omega_5\overset{2}{e}'_1\right) + \\
&+ I_5\dot{\omega}_5\overset{2}{e}'_5 + I_5\omega_5\left(\omega_3\overset{2}{e}'_6 + \omega_6\overset{2}{e}'_3 - \omega_1\overset{2}{e}'_4 - \omega_4\overset{2}{e}'_1\right) + \\
&+ I_6\dot{\omega}_6\overset{2}{e}'_6 + I_6\omega_6\left(-\omega_3\overset{2}{e}'_5 - \omega_5\overset{2}{e}'_3 - \omega_2\overset{2}{e}'_4 - \omega_4\overset{2}{e}'_2\right) = \\
&+ \left(-I_1\dot{\omega}_1 + I_2\omega_2\omega_3 - I_3\omega_3\omega_2 + I_4\omega_4\omega_5 - I_5\omega_5\omega_4\right)\overset{2}{e}'_1 + \\
&+ \left(-I_2\dot{\omega}_2 - I_1\omega_1\omega_3 + I_3\omega_3\omega_1 + I_4\omega_4\omega_6 - I_6\omega_6\omega_4\right)\overset{2}{e}'_2 + \\
&+ \left(-I_3\dot{\omega}_3 + I_1\omega_1\omega_2 - I_2\omega_2\omega_1 + I_5\omega_5\omega_6 - I_6\omega_6\omega_5\right)\overset{2}{e}'_3 + \\
&+ \left(\overset{2}{I}_4\dot{\omega}_4 + I_1\omega_1\omega_5 + I_2\omega_2\omega_6 - I_5\omega_5\omega_1 - I_6\omega_6\omega_2\right)\overset{2}{e}'_4 + \\
&+ \left(\overset{2}{I}_5\dot{\omega}_5 - I_1\omega_1\omega_4 + I_3\omega_3\omega_6 + I_4\omega_4\omega_1 - I_6\omega_6\omega_3\right)\overset{2}{e}'_5 + \\
&+ \left(\overset{2}{I}_6\dot{\omega}_6 - I_2\omega_2\omega_4 - I_3\omega_3\omega_5 + I_4\omega_4\omega_2 + I_5\omega_5\omega_3\right)\overset{2}{e}'_6
\end{aligned}
\tag{61}
$$

Therefore, Euler equations in the four-dimensional space take the following form:

$$
\begin{aligned}
I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + \left(I_4 - I_5\right)\omega_4\omega_5 - D_1 \\
I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3 + \left(I_4 - I_6\right)\omega_4\omega_6 - D_2 \\
I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + \left(I_5 - I_6\right)\omega_5\omega_6 - D_3 \\
I_4\dot{\omega}_4 &= \left(I_5 - I_1\right)\omega_1\omega_5 + \left(I_6 - I_2\right)\omega_2\omega_6 + D_4 \\
I_5\dot{\omega}_5 &= \left(I_1 - I_4\right)\omega_1\omega_4 + \left(I_6 - I_3\right)\omega_3\omega_6 + D_5 \\
I_6\dot{\omega}_6 &= \left(I_2 - I_4\right)\omega_2\omega_4 + \left(I_3 - I_5\right)\omega_3\omega_5 + D_6
\end{aligned}
\tag{62}
$$

In the case where all the components of the moment of force are equal to zero we obtain the following after multiplying by $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$, respectively:

$$
\begin{aligned}
I_1\dot{\omega}_1\omega_1 &= (I_2 - I_3)\omega_1\omega_2\omega_3 + \left(I_4 - I_5\right)\omega_1\omega_4\omega_5 \\
I_2\dot{\omega}_2\omega_2 &= (I_3 - I_1)\omega_1\omega_2\omega_3 + \left(I_4 - I_6\right)\omega_2\omega_4\omega_6 \\
I_3\dot{\omega}_3\omega_3 &= (I_1 - I_2)\omega_1\omega_2\omega_3 + \left(I_5 - I_6\right)\omega_3\omega_5\omega_6 \\
I_4\dot{\omega}_4\omega_4 &= \left(I_5 - I_1\right)\omega_1\omega_4\omega_5 + \left(I_6 - I_2\right)\omega_2\omega_4\omega_6 \\
I_5\dot{\omega}_5\omega_5 &= \left(I_1 - I_4\right)\omega_1\omega_4\omega_5 + \left(I_6 - I_3\right)\omega_3\omega_5\omega_6 \\
I_6\dot{\omega}_6\omega_6 &= \left(I_2 - I_4\right)\omega_2\omega_4\omega_6 + \left(I_3 - I_5\right)\omega_3\omega_5\omega_6
\end{aligned}
\tag{63}
$$

$$I_1\dot{\omega}_1\omega_1 + I_2\dot{\omega}_2\omega_2 + I_3\dot{\omega}_3\omega_3 + I_4\dot{\omega}_4\omega_4 + I_5\dot{\omega}_5\omega_5 + I_6\dot{\omega}_6\omega_6 = \qquad \textbf{(64)}$$

, which gives ultimately:

$$\frac{d}{dt}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + I_4\omega_4^2 + I_5\omega_5^2 + I_6\omega_6^2 \right) = 0 \qquad \textbf{(65)}$$

This equation leads to the law of conservation of kinetic energy for space channels:

$$\frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3} + \frac{J_4^2}{I_4} + \frac{J_5^2}{I_5} + \frac{J_6^2}{I_6} = const \qquad \textbf{(66)}$$

More generally:

$$\sum_{i=1}^{N_{n-2}} \frac{J_i^2}{I_i} = const \qquad \textbf{(67)}$$

Then, after multiplying by $I_1\omega_1, I_2\omega_2, I_3\omega_3, I_4\omega_4, I_5\omega_5, I_6\omega_6$, respectively, we obtain:

$$\begin{aligned}
I_1^2\dot{\omega}_1\omega_1 &= I_1(I_2 - I_3)\omega_1\omega_2\omega_3 + I_1\left(I_4 - I_5 \right)\omega_1\omega_4\omega_5 \\
I_2^2\dot{\omega}_2\omega_2 &= I_2(I_3 - I_1)\omega_1\omega_2\omega_3 + I_2\left(I_4 - I_6 \right)\omega_2\omega_4\omega_6 \\
I_3^2\dot{\omega}_3\omega_3 &= I_3(I_1 - I_2)\omega_1\omega_2\omega_3 + I_3\left(I_5 - I_6 \right)\omega_3\omega_5\omega_6 \\
I_4^2\dot{\omega}_4\omega_4 &= I_4\left(I_5 - I_1 \right)\omega_1\omega_4\omega_5 + I_4\left(I_6 - I_2 \right)\omega_2\omega_4\omega_6 \\
I_5^2\dot{\omega}_5\omega_5 &= I_5\left(I_1 - I_4 \right)\omega_1\omega_4\omega_5 + I_5\left(I_6 - I_3 \right)\omega_3\omega_5\omega_6 \\
I_6^2\dot{\omega}_6\omega_6 &= I_6\left(I_2 - I_4 \right)\omega_2\omega_4\omega_6 + I_6\left(I_3 - I_5 \right)\omega_3\omega_5\omega_6
\end{aligned} \qquad \textbf{(68)}$$

Ultimately, we obtain:

$$I_1^2\dot{\omega}_1\omega_1 + I_2^2\dot{\omega}_2\omega_2 + I_3^2\dot{\omega}_3\omega_3 + I_4^2\dot{\omega}_4\omega_4 + I_5^2\dot{\omega}_5\omega_5 + I_6^2\dot{\omega}_6\omega_6 = 0 \qquad \textbf{(69)}$$

, which gives

$$\frac{d}{dt}\left(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 + I_4^2\omega_4^2 + I_5^2\omega_5^2 + I_6^2\omega_6^2 \right) = 0 \qquad \textbf{(70)}$$

This equation leads to the law of conservation of angular momentum for space channels:

$$J_1^2 + J_2^2 + J_3^2 + J_4^2 + J_5^2 + J_6^2 = const \qquad \textbf{(71)}$$

, and more generally:

$$\sum_{i=1}^{N_{n-2}} J_i^2 = const \quad \textbf{(72)}$$

4.3 SPACE CHANNEL AT REST

Taking into consideration the symmetry of a space channel or, more properly, assuming that a space channel is symmetrical, we assume the following equalities:

$$I_{\breve{S}} \equiv I_1 = I_2 = I_3 \tag{73}$$

, and

$$I_{\breve{W}} \equiv I_4 = I_5 = I_6 \tag{74}$$

A similar assumption will be made with regard to the respective angular velocities:

$$\begin{aligned} \omega_{\breve{S}} &\equiv \omega_1 = \omega_2 = \omega_3 \\ \omega_{\breve{W}} &\equiv \omega_4 = \omega_5 = \omega_6 \end{aligned} \tag{75}$$

Therefore, directly from Formula **(48)**, we obtain a simplified formula of angular momentum for a space channel:

$$\overset{2}{J} = \overset{2}{I}\overset{2}{\omega} = -I_{\breve{S}}\omega_{\breve{S}}\left(\overset{2}{e_1}+\overset{2}{e_2}+\overset{2}{e_3}\right)+I_{\breve{W}}\omega_{\breve{W}}\left(\overset{2}{e_4}+\overset{2}{e_5}+\overset{2}{e_6}\right) \tag{76}$$

Additionally, for a space channel which is orthogonal to boundary hypersurfaces, we will assume that:

$$\omega_{\breve{W}} = 0 \tag{77}$$

After substituting in the angular momentum formula, we obtain:

$$\overset{2}{J} = \overset{2}{I}\overset{2}{\omega} = -I_{\breve{S}}\omega_{\breve{S}}\left(\overset{2}{e_1}+\overset{2}{e_2}+\overset{2}{e_3}\right) \tag{78}$$

In an interesting case, a space channel at rest only has a single component of angular momentum, for instance, component $e_1 \wedge e_4$:

$$\overset{2}{J}_{\breve{S}} = -I_{\breve{S}}\omega_1 e_1 \wedge e_4 \tag{79}$$

In fact, it appears that this type of a space channel is characterized by rotary motion around the axis of e_4 but also around the axis of e_1 while, in three-dimensional physics, rotations around the axis of e_1 are connected with the spin.

Moreover, let us note that there are exactly three space-channel orientations in which rotations around the axis of e_4 take place. Does it follow that quarks are space channels having such orientations? This question will be answered in the subsequent chapters.